AN ANALYTICAL SOLUTION FOR PARTIAL DIFFERENTIAL EQUATION GOVERNING VIBRATIONS OF A RECTANGULAR MEMBRANE.

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Abstract
One of the powerful analytical methods to solve partial differential equation is the Adomian Decomposition Method (ADM). In this paper, a general approach based on the generalized Fourier series expansion is applied to obtain an analytical solution. The solution is simplified in terms of a given orthogonal basis functions that these functions satisfy the boundary conditions. Based on the success of Fourier analysis and Hilbert space theory, orthogonal expansions undoubtedly count as fundamental concept of mathematical analysis. For the first time, We solved this equation using ADM and the results are compared with classical methods to demonstrate the accuracy of the scheme.

Keywords: Adomian Decomposition Method; Fourier series expansions; Orthogonal basis functions; Boundary value problems; Rectangular membrane; Membrane mathematical models.

INTRODUCTION

The study of membrane has attracted the interest of many scientific investigators over the years in view of its applications in science and engineering practices, especially in musical, materials, manufacturing and allied industries. Most scientific problems and phenomena in different fields of science and engineering occur nonlinearly. Adomian Decomposition Method provides the most versatile tool available in nonlinear analysis of scientific and engineering problems. This paper is devoted to the study of an analytical solution for partial differential equation governing vibrations of rectangular membrane using Adomian Decomposition Method. A general approach based on the generalized Fourier Series expansion is applied. The obtained analytical solution is simplified in terms of a given orthogonal basis functions that these functions satisfy the boundary conditions. Based on the success of Fourier analysis and Hilbert space theory, orthogonal expansions undoubtedly count as fundamental concept of mathematical analysis. For the first time, we solved this equation using ADM and the results are in excellent agreement with classical analysis.

Mathematical Formulation

Considering the vibrations of a uniform rectangular membrane for which due to uniformity, T and p are constants, been tension per cm length of the edge and ρ density per cm² of the membrane. Hence the appropriate governing equation is as follows:

\frac{\partial^2 U}{\partial t^2} = C^2 \left[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right] \quad (1)

Where L, Lx, Ly operators and inverse L⁻¹ are defined as follows:

L(U) = \frac{\partial^2 U}{\partial t^2}, \quad Lx(U) = \frac{\partial^2 U}{\partial x^2}, \quad Ly(U) = \frac{\partial^2 U}{\partial y^2} \quad (4)

\text{and } L^{-1} = \int_0^\sigma \int_0^t U(x,y,t) \, dt \, d\sigma \quad (5)

FORMULATION OF SOLUTION WITH ADOMIAN DECOMPOSITION METHOD.

Using the ADM, equation (1) can be rewritten in operator form as:

L(U) = C^2 \left[ Lx(U) + Ly(U) \right] \quad (3)

Where L, Lx, Ly operators and inverse L⁻¹ are defined as follows:

L(U) = \frac{\partial^2 U}{\partial t^2}, \quad Lx(U) = \frac{\partial^2 U}{\partial x^2}, \quad Ly(U) = \frac{\partial^2 U}{\partial y^2} \quad (4)

\text{and } L^{-1} = \int_0^\sigma \int_0^t U(x,y,t) \, dt \, d\sigma \quad (5)

ADM SOLUTION TO THE PROBLEM

In order to solve the equation using ADM, We used equation (3) and introduced the L⁻¹ operator on both sides of the equation, the solution of the equation can be written as:

U(x, y, t) = f(x, y) + tg(x, y) + (-1)^{i+1} \frac{C^2}{i!} \left[ Lx(U) + Ly(U) \right] \quad (6)

By using the ADM, U(x, y, t) can be expanded in terms of the U_i (x, y, t) components:

\sum_{i=0}^{\infty} U_i (x, y, t) \quad (7)

In order to find the components U_i (x, y, t), substitution of equation (7) into both sides of equation (6) yields:

\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} U_i (x, y, t) = f(x, y) + tg(x, y) \quad (8)

Considering the ADM, U_0 (x, y, t) is assumed to be of the following form: U_0 (x, y, t) = f(x, y) + tg(x, y) \quad (9)

Along with the following recurrence relation for U_i (x, y, t):

U_i (x, y, t) = (-1)^{i+1} \frac{C^2}{i!} \left[ Lx(U) + Ly(U) \right] \quad (10)

Thus the first t terms of the series are:

U_0 (x, y, t) = f(x, y) + tg(x, y) \quad (11)

U_1 (x, y, t) = (-1)^1 \frac{C^2}{1!} \left[ Lx(U) + Ly(U) \right] \quad (12)

U_2 (x, y, t) = (-1)^3 \frac{C^2}{3!} \left[ Lx(U) + Ly(U) \right] \quad (13)

U_i (x, y, t) = (-1)^i \frac{C^2}{i!} \left[ Lx(U) + Ly(U) \right] \quad (14)

Where, Lx = c^2 (Lx + Ly) \quad (15)

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Operator $L_n$ can be every operator according to relations. Infact, $f(x, y)$ and $g(x, y)$ in order in series are initial conditions.

$$U(x, y, 0) = f(x, y)$$

$$\frac{\partial U}{\partial t} (x, y, 0) = g(x, y)$$

It should be noted that the above functions $f(x, y), g(x, y)$ satisfy the boundary conditions of the problem. In addition, if all $U_i (x, y, t)$ functions satisfy the boundary conditions, then one may state that the sum of them also satisfies the boundary conditions. As shown in equation (14); $U_i (x, y, t)$ functions are determined by applying the (-1)$L_n$ operator to the functions $f(x, y), g(x, y)$. This may lead to $U_i (x, y, t)$ functions which either are zero or do not satisfy the boundary conditions at all. To prevent this difficulty, the function $f(x, y), g(x, y)$ are expanded in terms of known orthogonal functions $\varnothing_1 (x, y), \varnothing_2 (x, y), \ldots$, as a generalization of the Fourier Series expansion. $\varnothing_1 (x, y), \varnothing_2 (x, y), \ldots$ can be selected to satisfy the boundary conditions before and after applying (-1) $L_n$ operator. As a result, the function $f(x, y)$ $g(x, y)$ becomes:

$$f(x, y) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} \varnothing_{kj} (x, y)$$

$$g(x, y) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} b_{kj} \varnothing_{kj} (x, y)$$

Where the coefficients $a_{kj}, b_{kj}$ of Fourier expansion are given by the following relations:

$$a_{kj} = \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \varnothing_{kj} (x, y) \, dy \, dx$$

$$b_{kj} = \int_{x=0}^{a} \int_{y=0}^{b} g(x, y) \varnothing_{kj} (x, y) \, dy \, dx$$

The best set of functions for the generalized Fourier expansion in the case of our physical problem is the set of eigenfunctions of the following self – adjoint system:

$$L_n \varnothing_{kj} (x, y) = \lambda_{kj} \varnothing_{kj} (x, y)$$

Previous studies indicate that the eigenvalue problem defined in equation (24) yields an infinite set of real eigenvalues and eigenfunctions $[\lambda_{kj}, \varnothing_{kj} (x, y)]$. These eigenfunctions constitute the basis for infinite dimensional Hilbert space. Therefore, every function $h(x, y)$ with continuous $L_n h(x, y)$ which satisfies the boundary conditions of the system that can be expanded in an absolutely and uniformly convergent series in the eigenfunctions. According to equation (24), we can normalize the eigenfunctions as follows:

$$\int_{x=0}^{a} \int_{y=0}^{b} \varnothing_{h} (x, y) \, \varnothing_{h} (x, y) \, dy \, dx = \begin{cases} 1 & \text{lh} \ne k_j \\ 0 & \text{lh} = k_j \end{cases}$$

$$\int_{x=0}^{a} \int_{y=0}^{b} \varnothing_{h} (x, y) \, L_n \varnothing_{h} (x, y) \, dy \, dx = \frac{0}{\lambda_{kj}}$$

And finally we obtained:

$$L_n \varnothing_{h} = \frac{1}{\lambda_{kj}} \varnothing_{h}$$

Using equation (18) and (19) and substituting equations (20) and (21) into equation (14) and embedding the result into equation (8) yields:

$$U(x, y, t) = \sum_{i=0}^{\infty} U_i (x, y, t)$$

$$f(x, y) + t g(x, y) + \sum (-1)^i [c^2 (L_i + L_i^*)] \left. \phi(x, y) + t \sum_{i=1}^{\infty} g(x, y) \right|_{i=0}^{(i+1)!}$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} \varnothing_{kj} (x, y) + t \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} b_{kj} \varnothing_{kj} (x, y) + \sum (-1)^i \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{i=1}^{\infty} \frac{t^i}{i!}$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} [a_{kj} \sum (-1)^i \lambda_{kj} \varnothing_{kj} + b_{kj} \sum (-1)^i \frac{t^i}{i!} \lambda_{kj} \varnothing_{kj}]$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} [a_{kj} \sum (-1)^i \lambda_{kj} \varnothing_{kj} + b_{kj} \sum (-1)^i \frac{t^i}{i!} \lambda_{kj} \varnothing_{kj}]$$

$$U(x, y, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} [a_{kj} \cos (c_{kj} t) + b_{kj} \sin (c_{kj} t)] \varnothing_{kj} (x, y)$$

**VALIDITY OF THE ADOMIAN DECOMPOSITION METHOD**

The validity of the scheme is investigated in this section. The results are compared with known exact solutions. According to equation (25) – (28), the solution in this case, the generalized Fourier series expansion functions are determined that these functions satisfy the boundary conditions before and after applying (-1)$L_n$ operator. The corresponding differential equation has the same solution as shown below. The orthogonal basis eigenfunctions expansion that satisfy this problem are:

$$\varnothing_{h} (x, y) = \sin (ktx) \sin (lty)$$

and the corresponding eigen values are:

$$\lambda_{ij} = \pi^2 k^2 + \pi^2 l^2$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This solution is exactly the same as the solutions of previous studies in literature.

**CONCLUSION**

In this paper, ADM has been successfully applied to solve the problem. The results obtained by the ADM are in excellent agreement with classical methods. But using the ADM is based upon applications of Fourier series expansion and the orthogonal basis functions expansion in Hilbert space, so developing the method for different applications is not easy and finding these orthogonal functions are also difficult yet possible.

**References**


