SOLUTION FOR PARTIAL DIFFERENTIAL EQUATION GOVERNING
RECTANGULAR UNIFORM FLEXIBLE PLATE.

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Abstract
One of the powerful analytical methods to solve partial differential equations is the Adomian Decomposition Method (ADM). In this paper, a general approach based on the generalized Fourier series expansion is applied. The obtained analytical solution is simplified in terms of a given orthogonal basis function that these functions satisfy the boundary conditions of the plate. The best set of functions for the generalized Fourier expansion in the case of our physical problem is the set of Eigen functions of the self-adjoint system. Previous studies indicate that the eigenvalue problem yields an infinite set of real eigenvalues and Eigen functions. These functions constitute the basis for the infinite-dimensional Hilbert space. Therefore, every function which is continuous satisfies the boundary conditions of the system that can be expanded in an absolutely and uniformly convergent series in the Eigen functions and these functions are normalized. In this paper, the generalized Fourier series expansion functions are determined that these functions satisfy the boundary conditions before and after applying the LM operator to the functions which either are zero or do not satisfy the boundary conditions at all. To prevent this difficulty, the functions are expanded in terms of known orthogonal functions and these functions are selected to satisfy the boundary conditions before and after applying LM operator. For the first time, we solved this problem using ADM and compared the result with those of the classical analysis to demonstrate the validity of the scheme.

Keywords: Adomian Decomposition Method; fourth-order PDE and flexible plate; Fourier series expansions; orthogonal basis functions expansion.

INTRODUCTION

The study of flexible plate has attracted the interest of many scientific investigators over the years in view of its applications in civil and mechanical engineering sciences, for design of slabs and system with plate behaviors. Most scientific problems and phenomena in different fields of science and engineering occur non-linearly. Adomian Decomposition Method provides the most versatile tool available in nonlinear analysis of scientific and engineering problems.

This paper is devoted to the study of homogeneous fourth-order parabolic partial differential equation for rectangular uniform flexible plate using Adomian decomposition method. A general approach based on generalized Fourier series expansion is used. The obtained analytical solution is simplified in term of a given orthogonal basis functions that these functions satisfy the boundary conditions. For the first time, we solved this problem using ADM and the results are in excellent agreement with classical methods.

MATHEMATICAL FORMULATION OF THE PROBLEM.

The homogeneous fourth-order parabolic partial differential equation, governing rectangular uniform flexible plate is:

\[
\begin{align*}
D \left[ \psi_4 \right] &= n_x \frac{\partial^4 u}{\partial x^4} + n_y \frac{\partial^4 u}{\partial y^4} + 2n_{xy} \frac{\partial^4 u}{\partial x \partial y} - \\
\rho h \frac{\partial^4 u}{\partial t^4} &= ... ... ... (1)
\end{align*}
\]

This is a variable co-efficient homogenous fourth-order parabolic partial differential equation, where:

\[ u = (x, y, t) \] the deflection middle surface of plate, \( D \) is plate bending stiffness, \( \rho \) is the plate material density, \( n_x, n_y, n_{xy} \) in order in series are in-plane forces in parallel to \( x, y \) direction and \( h \) is the plate thickness.

The homogeneous boundary condition (HBCS) for rectangular plate area combination as the following (for \( x \)-direction and at \( x = a \))

Two HBCS for free edge (at \( x = a \))

\[
\begin{align*}
[2 - v] \frac{\partial^2 u}{\partial y \partial x^2} &= 0 \\
\rho h \frac{\partial^2 u}{\partial y^2} &= 0 \quad ... ... ... (2)
\end{align*}
\]

And
Two HBCS for fixed edge (at x=p) \(\frac{\partial u}{\partial x} = 0\) \hspace{1cm} (3)

Or

Two HBCS for simple edge (at \(x=p\))

\[\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial y^2} & = 0 \\
\end{align*}\]

Where \(v\) the Poisson ratio of plate material.

It should be noted that two HBCs must be satisfied at \(x = a, x = p\), and two HBCs must also be satisfied at \(y = 0, y = q\).

In this paper, the solution of the governing equation of a uniform flexible plate is presented which takes the boundary conditions of the problem into account. The solution is compared with those of classical methods to demonstrate the validity of the scheme.

**Formulation with ADM**

Using the ADM for EQ. (1) can be rewritten in operator form as:

\[\rho h[L_t]u + D[L_{xy}]u = n_x[L_x]u + n_y[L_y]u + 2n_{xy}[L_{xy}]u\]

Where the \(L_t, L_{xy}^4, L_x, L_y, L_{xy}\) operators and \(L_t^{-1}\) inverse are defined as follows:

\[L_t u = \frac{\partial^2 u}{\partial x^2}, \quad L_y u = \frac{\partial^2 u}{\partial y^2}, \quad L_{xy} u = \frac{\partial^2 u}{\partial x \partial y}\]

And

\[L_t^{-1}(u) = \int_0^t \int_0^\tau u(x,y,t) \, dt \, ds\] (7)

In order to solve the equation, using ADM, we introduce the \(L_t^{-1}\) operator on both sides of eq. (5), the solution of the equation can be written as:

\[L_t^{-1}(u) = \frac{-D}{\rho h}[L_{xy}]u + \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

\[u(x,y,t) = \int_0^t \int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

By using the Adomian decomposition method \(u(x,y,t)\) can be expanded as an infinite series expansion in terms of the \(u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t)\) \hspace{1cm} (10)

In order to find the components, \(u_i(x,y,t)\) substitution of Eq. (10) into both sides of Eq (9) yields:

\[\sum_{i=0}^{\infty} u_i(x,y,t) = f(x,y) + t g(x,y) + \left((-\frac{1}{\rho h})L_t^{-1}[D(L_{xy}^4 + n_xL_x - n_yL_y - 2n_{xy}L_{xy})]\sum_{i=0}^{\infty} u_i(x,y,t)\right) \hspace{1cm} (11)

Considering the decomposition methods, \(u_0(x,y,t)\) is assumed to be of the following form:

\[u_0(x,y,t) = f(x,y) + tg(x,y) \hspace{1cm} (12)

Along with the following recurrence relation for \(u_i(x,y,t)\):

\[u_1(x,y,t) = \left((-\frac{1}{\rho h})L_t^{-1}[D(L_{xy}^4 + n_xL_x - n_yL_y - 2n_{xy}L_{xy})]\right) u_0(x,y,t)\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

Thus the first \(i\) terms of the series are:

\[u_0(x,y,t) = f(x,y) + tg(x,y) \hspace{1cm} (14)

\[u_1(x,y,t) = \left((-\frac{1}{\rho h})L_t^{-1}[D(L_{xy}^4 + n_xL_x - n_yL_y - 2n_{xy}L_{xy})]\right) u_0(x,y,t)\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]

\[\int_0^\tau f(x,y,t) + t g(x,y) + (-1)^i L_t^{-1}\frac{1}{(2i)!}[L_{xy}]u \frac{n_x}{\rho h}[L_x]u + \frac{n_y}{\rho h}[L_y]u + 2\frac{n_{xy}}{\rho h}[L_{xy}]u\]
\[ u(x,y,t) = \left\{ (-1)^i L_m^i \right\} \int_{(2i)} f(x,y) + \left( \frac{t^{2i+1}}{(2i+1)!} g(x,y) \right) \] (17)

Where

\[ L_m = \left\{ \left( \frac{D}{\rho h} \right) L_{qy}^4 - \frac{n_x}{\rho h} L_x - \frac{n_y}{\rho h} L_y - 2\frac{n_{xy}}{\rho h} (L_{xy}) \right\} \] (18)

\[ L_m^i = \left\{ \left( \frac{D}{\rho h} \right) L_{qy}^4 - \frac{n_x}{\rho h} L_x + \left( \frac{-n_y}{\rho h} \right) L_y + \left( \frac{-2 n_{xy}}{\rho h} \right) L_{xy} \right\} \] (19)

\[ L_p^i = L_p \left( \frac{L_{qy}^4}{\rho h} \right) \] (20)

Operator \( L_p \) can be every operator according to relations. Infact, \( f(x,y), g(x,y) \) in order in series are initial displacement and velocity of plate and then:

\[ u(x,y,t) = f(x,y) \] (21)

\[ \frac{\partial u}{\partial t} (x,y) = g(x,y) \] (22)

It should be noted that the above functions \( f(x,y) \) and \( g(x,y) \) satisfy the boundary conditions of the problem. If all \( u_i(x,y,t) \) functions satisfy the boundary conditions, then the sum of them also satisfies the boundary conditions. As shown in equation (17):

\[ u_i(x,y,t) \] functions are determined by applying the \( (-1)^i L_m^i \) operator to the functions which either are Zero or do not satisfy the boundary conditions all. To prevent this difficulty function \( f(x,y) \), \( g(x,y) \) are expanded in terms of known orthogonal functions \( \varphi_i(x,y) \), \( \varphi_2(x,y) \) ... as a generalization of the Fourier series expansion \( \varphi_1(x,y) \), \( \varphi_2(x,y) \) ... can be selected to satisfy the boundary conditions before and after applying \( (-1)^i L_m^i \) operator. As a result, functions \( f(x,y) \), \( g(x,y) \) become:

\[ f(x,y) = \sum_{n=1}^{\infty} a_{nm} \varphi_{nm}(x,y), \ldots \] (23)

\[ g(x,y) = \sum_{n=1}^{\infty} b_{nm} \varphi_{nm}(x,y), \ldots \] (24)

Where the coefficients \( a_{nm} \), \( b_{nm} \) are given by the following relations:

\[ a_{nm} = \int_{x=0}^{p} \int_{y=0}^{q} f(x,y) \varphi_{nm}(x,y) dy dx \] (25)

\[ a_{nm} = \int_{x=0}^{p} \int_{y=0}^{q} g(x,y) \varphi_{nm}(x,y) dy dx \] (26)

The best set of functions for the generalized Fourier expansion in the case of our physical problem is the set of eigenfunctions of the following self-ad joint system:

\[ L_m \varphi_{nm}(x,y) = \lambda_{nm} \varphi_{nm}(x,y) \] (27)

Previous studies indicate that the eigenvalue problem defined in equation (27) yields an infinite set of real eigenvalues and eigenfunctions \( (\lambda_{nm}, \varphi_{nm}(x,y)) \). These eigenfunctions constitute the basis for the infinite-dimensional Hilbert space.

Therefore, every function \( h(x,y) \) with continuous \( L_m \) which satisfies the boundary conditions of the system that can be expanded in an absolutely and uniformly convergent series in the eigenfunctions. According to equation (27), we can normalize the eigenfunctions as follows:

\[ \int_{x=0}^{p} \int_{y=0}^{q} \varphi_{nm}(x,y) \varphi_{nm}(x,y) dy dx = \sum_{n=1}^{\infty} \lambda_{nm} \] (28)

\[ \int_{x=0}^{p} \int_{y=0}^{q} \varphi_{nm}(x,y) L_m \varphi_{nm}(x,y) dy dx = \sum_{n=1}^{\infty} \lambda_{nm} \] (29)

And finally we obtain:

\[ L_m^i \varphi_{nm} = (\lambda_{nm}) \varphi_{nm} \] (30)

Using equation (21) and (22) and substituting equations (23) and (24) into equation (17) and embedding the result into equation (11) yields:

\[ u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \varphi_{nm}(x,y) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \varphi_{nm}(x,y) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \left( \frac{D}{\rho h} \right) L_q^4 - \frac{n_x}{\rho h} L_x + \left( \frac{-n_y}{\rho h} \right) L_y + \left( \frac{-2 n_{xy}}{\rho h} \right) L_{xy} \right) \frac{t^{2i}}{(2i+1)!} f(x,y) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \left( \frac{D}{\rho h} \right) L_q^4 - \frac{n_x}{\rho h} L_x + \left( \frac{-n_y}{\rho h} \right) L_y + \left( \frac{-2 n_{xy}}{\rho h} \right) L_{xy} \right) \frac{t^{2i+1}}{(2i+1)!} g(x,y) \] (31)
A rectangular plate is considered with constant D is the plate bending stiffness, \( \rho \) is plate material density and \( h \) is the plate thickness, which is simply supported at each four edges. The corresponding differential equation is:

\[
D \left( \nabla^4 u (x, y, t) + \frac{\rho h}{2} \frac{\partial^2 u(x,y,t)}{\partial t^2} \right) = 0
\]

With boundary conditions as in Ref [8,9]

\[
\frac{\partial^2 u(0,y,t)}{\partial x^2} = 0,
\]

\[
\frac{\partial^2 u(p,y,t)}{\partial x^2} = 0,
\]

\[
\frac{\partial^2 u(x,0,t)}{\partial y^2} = 0,
\]

\[
\frac{\partial u(x,q,t)}{\partial y^2} = 0.
\]

According to the equations (28), (29) and (31), generalized Fourier series expansion functions are determined that these functions satisfy the boundary conditions before and after applying the \( L_M \) operator.

Therefore, \( L_M = \frac{D}{\rho h} L_V \),

\[
\varphi_{nm}(x, y) = \frac{2}{\sqrt{\pi q}} \sin \left( \frac{m \pi x}{p} \right) \sin \left( \frac{n \pi y}{q} \right) 
\]

and

\[
\sqrt{\lambda_{nm}} = \pi^2 \left[ \left( \frac{m}{p} \right)^2 + \left( \frac{n}{q} \right)^2 \right] \left[ \frac{D}{\rho h} \right]
\]

This solution is exactly the same as the solution of Ref [8,9] and other previous studies in literature.

**CONCLUSION**

In this paper, ADM has been successfully used to obtain the plates equation. The results obtained by decomposition method are in excellent agreement with classical method. But using the ADM is based upon the orthogonal functions, so developing the method for different applications is not easy and finding these orthogonal functions are also difficult but possible.

**References**